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SUBSTITUTION TILINGS WITH STATISTICAL CIRCULAR SYMMETRY

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ABSTRACT. Two new series of substitution tilings are introduced in which the tiles appear in infinitely many orientations. It is shown that several properties of the well-known pinwheel tiling do also hold for these new examples, and, in fact, for all substitution tilings showing tiles in infinitely many orientations.

Dedicated to my teacher Ludwig Danzer on the occasion of his 80th birthday

1. **Introduction**

In this article, we introduce two new series of nonperiodic substitution tilings in the plane, where the tiles appear in infinitely many different orientations. There seems to be a growing interest in such objects, cf. [ORS], [SA2], [MPS], [BFG], [Y]. The standard example is certainly the pinwheel tiling of Conway and Radin [RA1], and most work about tilings with tiles in infinitely many orientations is dedicated to this special example. It stimulated several re-formulations of concepts which play a role in the theory of nonperiodic tilings. For instance, it suggests to define the dynamical system of such a tiling $\mathcal{T}$ as the closure (in an appropriate topology) of the orbit of $\mathcal{T}$ under the action of the Euclidean group $E(2)$ of $\mathbb{R}^2$, rather than the translation group $\mathbb{R}^2$ alone.

Further examples of tilings showing tiles in infinitely many orientations are rarely found in the literature. Sadun gave a generalization of the pinwheel tiling [SA1], yielding a countable number of different substitution tilings with this property. Apart from this, there are only few examples known to the author. One is folklore, but nevertheless not widely known. It is shown in Figure 1. Two more were found by Harriss [FH]. The tilings introduced in Sections 4 and 5 of the present article show that the occurrence of infinitely many orientations in substitution tilings is not necessarily a rare effect. These tilings include examples of finite local complexity as well as infinite local complexity (w.r.t. Euclidean motions), with an arbitrary number of prototiles, and with or without primitive substitution matrices. Some of the occurring substitution factors are Pisot-Vijayaraghavan numbers (including the smallest one), others are not.

Section 2 states some basic definitions and facts about substitution tilings. In Section 3, we define statistical circular symmetry of a substitution tiling and prove a technical result which turns out to be useful in the sequel. Section 6 is dedicated to prove that many properties of the pinwheel tiling generalise to the whole class of substitution tilings with tiles in infinitely many orientations. In particular, these properties are uniform distribution of orientations, uniform patch frequencies (w.r.t. to the topology used in [ORS] as well as the local rubber topology in [BL]), circular symmetry of the autocorrelation, and therefore of the diffraction spectrum [MPS].

2. **Substitution Tilings**

In the following, $B_r(x)$ denotes the closed ball of radius r around x. A rotation through an angle θ about the origin is denoted by R_θ. Throughout the text, we will identify the Euclidean plane with the complex plane, choosing freely the point of view which fits better to the question at hand.

A *tiling* of $\mathbb{R}^d$ is a covering of $\mathbb{R}^d$ with compact sets — the *tiles* — which is also a packing of $\mathbb{R}^d$. A tiling $\mathcal{T}$ is *nonperiodic* if $\mathcal{T} + t = \mathcal{T}$ implies $t = 0$.

Tile substitutions are a simple and powerful tool to generate interesting nonperiodic tilings. The basic idea is to give a finite set of building blocks — the *prototiles* — together with a rule how to enlarge each prototile and then dissect it into copies of the original prototiles, compare Figures 1, 4, 5. Although the concept applies to arbitrary dimensions, we restrict ourselves in the following to tilings in the plane $\mathbb{R}^2$, in order to keep the notation simple.

Formally, a *substitution* σ in $\mathbb{R}^2$ is defined for a collection of prototiles $T_1, \ldots, T_m$ by $\sigma(T_i) = \{\varphi_{ijk}(T_j) \,|\, \varphi_{ijk} \in \Phi_{ij}, j = 1 \ldots m\}$, where Φ_{ij} $(1 \le i, j \le m)$ are sets (possibly empty) of affine maps from $\mathbb{R}^2$ to $\mathbb{R}^2$. Usually, the maps are of the form $x \mapsto R_\alpha x + t$ for some $\alpha \in [0, 2\pi[$, $t \in \mathbb{R}^2$. It is a matter of taste whether one allows also reflections or not. For convenience, we will switch between the two concepts. The particular choice will always be obvious from the context. Two tiles $T, T' \in \mathcal{T}$ are *of the same type*, if they are congruent to the same prototile T_i. (Sometimes one has to consider tiles which are congruent but of different types. Then, each tile will get a label, assigning its type to it. But here, we don't need to consider such cases.)

Definition 2.1. *A substitution σ is called* self-similar*, if there is some $\lambda > 1$ such that for all prototiles T_i:*

$$\lambda T_i = \bigcup_{T \in \sigma(T_i)} T$$

Then, λ is called the substitution factor*.*

Synonyms of substitution factor are *inflation factor* or *length expansion*. In the following we consider self-similar substitutions only. The substitution σ extends in a natural way to all collections of copies of prototiles: Note that any such collection in the plane — and in particular, each tiling — can be represented as $\{R_{\alpha_1} T_{i_1} + t_1, R_{\alpha_2} T_{i_2} + t_2, \ldots\}$, where the R_{α_i} are rotations, $t_i \in \mathbb{R}^2$. The image of this set under σ is defined as $\{R_{\alpha_1} \sigma R_{\alpha_1}^{-1} T_1 + \lambda t_1, R_{\alpha_2} \sigma R_{\alpha_2}^{-1} T_2 + \lambda t_2, \ldots\}$, where λ is the substitution factor. In particular, if the set represents a tiling, this defines the action of σ on a tiling.

For such a tiling $\mathcal{T} = \{R_{\alpha_1}(T_{i_1}) + t_1, R_{\alpha_2}(T_{i_2}) + t_2, \ldots\}$, we assign an angle $\alpha(T)$ to each tile $T = R_\alpha T_i + t$ in $\mathcal{T}$ by

$$\alpha(T) = \alpha. \tag{1}$$

The following definition helps us to avoid certain pathological cases. A *patch* is a finite subset of a tiling $\mathcal{T}$, that is, a finite set of tiles in $\mathcal{T}$.

Definition 2.2. *Let σ be a substitution with prototiles $T_1, \ldots, T_m$ as defined above. A tiling $\mathcal{T}$ is called* substitution tiling *(w.r.t. to the substitution σ), if each patch in $\mathcal{T}$ is congruent*

to some patch in $\sigma^n(T_i)$ *for appropriate* n, i.
The set of all substitution tilings w.r.t. σ *is called the* tiling space $\mathbb{X}_\sigma$.

A useful object is the matrix $S_\sigma := (|\Phi_{ij}|)_{1\le i,j\le m}$. A substitution σ is called *primitive*, if S_σ is primitive. Recall: A nonnegative matrix M is called primitive if there is some $k \ge 1$ such that M^k contains positive entries only. By Perron's theorem ([Per], see also [Se]), each primitive nonnegative matrix has a unique eigenvalue which is real, positive, and larger than all other eigenvalues in modulus. It is not hard to see that the substitution factor of a primitive self-similar tiling in $\mathbb{R}^2$ is the square root of the Perron-eigenvalue of the substitution matrix, see for instance [F].

3. Statistical circular symmetry

The following definition helps to simplify terminology.

Definition 3.1. *A substitution tiling* $\mathcal{T}$ *is called* pinwheel-like*, if there are infinitely many different values* $\alpha(T)$ *for* $T \in \mathcal{T}$*, with* $\alpha(T)$ *defined as in* (1).

For a primitive substitution tiling, this is equivalent to requiring that all copies of each certain prototile occur in infinitely many orientations in $\mathcal{T}$.

It is known that the pinwheel tiling fulfils an even stronger condition, namely, the orientations $\alpha(T)$ of the tiles are uniformly distributed in $[0, 2\pi[$, see [Ra1], [MPS]. Recall that a sequence $(\alpha_j)_{j\ge1}$ is called *uniformly distributed* in $[0, 2\pi[$, if, for all $0 \le x < y < 2\pi$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} 1_{[x,y]}(\alpha_j) = \frac{x-y}{2\pi}.$$

Following [MPS], we will call this property *statistical circular symmetry*. However, it requires some care to define this properly. Since the sum above is not absolutely convergent, the order of the elements in the sequence matters. Therefore, one needs to specify how to arrange the tiles $T_i \in \mathcal{T}$ in a sequence $(T_i)_{i\ge0}$.

Definition 3.2. *Let* $\mathcal{T}_\sigma = \{T_1, T_2, \ldots\}$ *be a primitive substitution tiling, such that the sequence* $(T_j)_{j\ge1}$ *satisfies the following: for all* $n \ge 1$*, there is some* $\ell \ge n$ *such that the patch* $\{T_1 \ldots, T_\ell\}$ *is congruent to* $\sigma^k(T_i)$ *for some* k, i*. The tiling* $\mathcal{T}_\sigma$ *has* statistical circular symmetry*, if, for all* $0 \le x < y < 2\pi$*, one has:*

$$\lim_{r\to\infty} \frac{1}{n} \sum_{j=1}^{n} 1_{[x,y]}(\alpha(T_j)) = \frac{x-y}{2\pi}.$$

In plain words, the definition ensures that there are infinitely many n such that $T_1, \ldots, T_n$ are exactly the tiles contained in some *supertile* $\sigma^k(T_i)$. Although this definition seems rather technical, it ensures that tiles (resp. the associated angles) are ordered in a natural way. With respect to the limit, for instance, it is equivalent to ordering tiles according to their distance from the origin in an increasing order. This fact follows from standard properties of primitive substitution tilings.

Since the pinwheel tilings are not only pinwheel-like, but also of statistical circular symmetry, one may argue that the term 'pinwheel-like' is not well chosen. However, Theorem 6.1 below

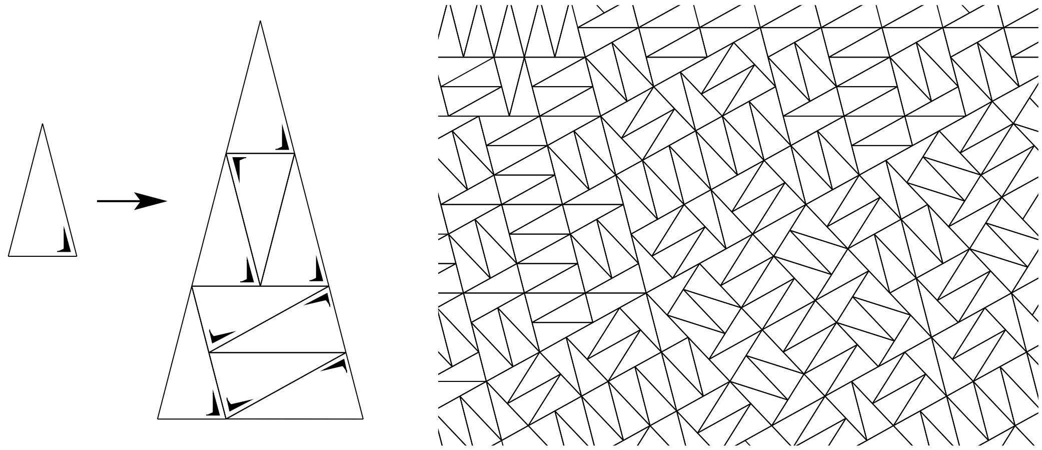

FIGURE 1. A simple substitution rule which generates tilings of statistical circular symmetry.

shows the equivalence of pinwheel-likeness and statistical circular symmetry for primitive substitution tilings, so the term pinwheel-like becomes obsolete.

The substitution rule shown in Figure 1 defines a family of pinwheel-like tilings. This example is somehow folklore. It was communicated to the author by L. Danzer [DAN1]. The substitution factor is 3, there is only one prototile T — an isosceles triangle with edge lengths 1, 2 and 2 — and the substitution uses only direct congruences, not reflections. The prototile T is mirror-symmetric along its vertical axis. Therefore, we equip it with a mark in order to illustrate the absence of reflections in the substitution rule. The substituted triangle $\sigma(T)$ contains four copies of T in the same orientation as T, and two copies of T which are rotated by $\theta = -\arccos(1/4)$. It is known that $\arccos(1/n) \notin \pi\mathbb{Q}$ for all $n \geq 3$. By Proposition 3.4 below, the tilings in this example are pinwheel-like.

During the last years it, turned out that it is fruitful to consider tiling spaces or *hulls* of tilings rather than particular single tilings. The *hull* of a tiling $\mathcal{T}$ in $\mathbb{R}^d$ is the closure of the set $\{\mathcal{T} + x \,|\, x \in \mathbb{R}^d\}$ in the local topology, which is given by the following metric [SO]. Let

$$\widetilde{d}(\mathcal{T}, \mathcal{T}') = \inf_{\varepsilon > 0} \{\varepsilon \,|\, \mathcal{T} \cap B_{1/\varepsilon} = R_\theta \mathcal{T}' + t, \, \|t\| \leq \varepsilon, \, |\theta| \leq \varepsilon\},$$

and

$$d(\mathcal{T}, \mathcal{T}') = \min\{\frac{1}{\sqrt{2}}, \widetilde{d}(\mathcal{T}, \mathcal{T}')\}.$$

In plain words, two tilings are close, if they agree on a large ball (of radius $1/\varepsilon$) around the origin, after a small rotation R_θ through θ, and after a small translation t. In the case of tilings where all tiles occur only in finitely many orientations, the rotation part R_θ is usually omitted. Then, one distinguishes tiles not up to congruence, but up to translations; the number of prototiles stays finite. We should mention that, in the case of tilings with finite local complexity (see Section 7), this topology is the same as the local rubber topology [BL] and the local topology [SC]. For brevity, we don't go into details here.

The hull of a tiling in $\mathbb{R}^2$, together with the action of the Euclidean group $E(2) = \mathbb{R}^2 \rtimes O(2)$, is a dynamical system $\big(\mathbb{X}_\mathcal{T}, E(2)\big)$. In this article we consider primitive substitution tilings only. Note that primitivity ensures that all types of prototiles occur in some $\sigma^k(T_i)$, for all i

and k large enough. This property is a key for the following theorem, which is a variant of Gottschalk's theorem, see for instance [Rob].

Theorem 3.3. *If σ is a primitive substitution, the hull $\mathbb{X}_{\mathcal{T}}$ of each substitution tiling $\mathcal{T}$ in $\mathbb{X}_\sigma$ is $\mathbb{X}_\sigma$ itself. Equivalently, the dynamical system $\big(\mathbb{X}_{\mathcal{T}}, E(2)\big)$ is minimal.*

Consequently, in the case of primitive substitutions, the dynamical system $\big(\mathbb{X}_\sigma, E(2)\big)$ has a unique meaning.

The following result is needed for the proofs of Theorems 6.1 and 6.2. One direction ('if') is well-known, the other direction seems to be new and is necessary for the proof of Theorem 6.1.

Proposition 3.4. *Let σ be a primitive substitution in $\mathbb{R}^2$ with prototiles $T_1, \ldots, T_m$. Each substitution tiling $\mathcal{T}_\sigma$ is pinwheel-like, if and only if there are n, i such that $\sigma^n(T_i)$ contains tiles T, T' of the same type, where $\theta = \alpha(T) - \alpha(T') \notin \pi\mathbb{Q}$.*

Proof. Without loss of generality, let $\alpha(T) = \theta$ and $\alpha(T') = 0$. Here, T and T' are no mirror images of each other. If there are prototiles which are mirror symmetric, we break the symmetry with some markings to avoid ambiguities.

Since σ is primitive, there is some k such that a tile of type T_i is contained in $\sigma^k(T)$ as well as in $\sigma^k(T')$. These two tiles are also rotated against each other by θ. Thus, $\sigma^{n+k+n}(T_i)$ contains two copies of T which are rotated against each other by 2θ. It follows inductively that there are copies of T rotated against each other by $n\theta$ for all $n \geq 0$. Since $\theta \notin \pi\mathbb{Q}$, the values $n\theta \mod \pi$ are pairwise different. Thus the tiles of type T occur in infinitely many orientations. Since σ is primitive, the same is true for all prototiles T_j.

For the other direction, let $\mathcal{T}_\sigma$ be pinwheel-like. We proceed by showing that if there are infinitely many different angles $\theta = \alpha(T) - \alpha(T')$ as above, at least one of them is irrational. Consider the finitely many angles $\alpha_1, \ldots, \alpha_n$ occurring in the rotational parts of maps in the definition of σ. Then, all angles $\alpha(T)$ occurring in $\mathcal{T}$ are linear combinations

$$\Big(\sum_{i=1}^{n} \lambda_i \alpha_i\Big) \mod 2\pi \qquad (\lambda_i \in \mathbb{Z}),$$

in other words, they are elements of the finitely generated $\mathbb{Z}$-module $\langle \alpha_1, \ldots, \alpha_n \rangle$. If all α_i are rational angles, we are done, since then there are only finitely many such linear combinations mod π, thus only finitely many angles $\alpha(T)$. If some α_i are irrational, it is slightly more complicated:

In order to get rid of the common factor π, let $\beta_i = \frac{\alpha_i}{2\pi}$. Consider the $\mathbb{Z}$-module $M = \langle \beta_1, \ldots, \beta_n \rangle_{\mathbb{Z}}$. Since it is finitely generated and torsion-free (the only element of finite order is 0), there is a basis $\gamma_1, \ldots, \gamma_k$ of M, compare [Rot, Theorem 10.19]. Let m be the smallest positive integer in M. (If there is no such number, then all differences $\beta_i - \beta_j$ $(j \neq i)$ are irrational, and we are done.) It has a unique representation $m = \sum_{i=1}^{k} \lambda_i \gamma_i$. All other integers ℓ in M are integer multiples of m. (Otherwise there is a positive integer $\lambda\ell + \mu m < m$). Let $q = \gcd(\lambda_1, \ldots, \lambda_k)$. Then $\frac{1}{q}$ is the smallest positive rational number in M, and all others are of the form $\frac{p}{q}$, $p \in \mathbb{Z}$. In particular, there are only finitely many rational numbers mod 1 in M, hence only finitely many rational angles $\theta = \alpha(T) - \alpha(T') \in \pi\mathbb{Q}$.

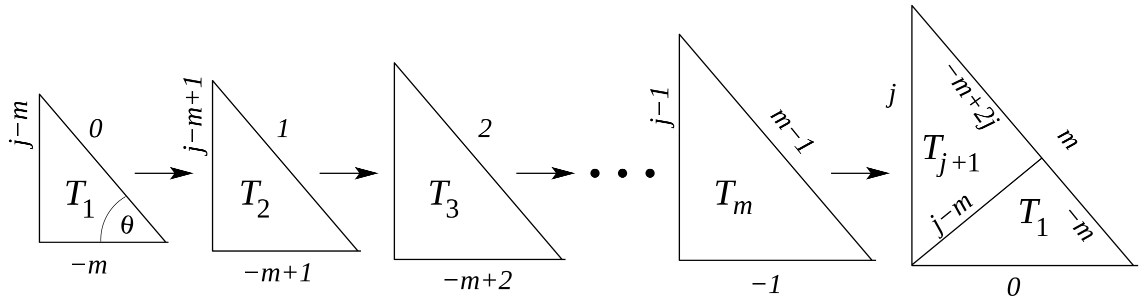

FIGURE 2. The Pythagoras substitution. The edge labels give the edge lengths in terms of powers of λ. For instance, $-m$ is to be read as λ^{-m}. In particular, 0 means $\lambda^0 = 1$.

But by the definition of pinwheel-likeness there are infinitely many such θ. Thus at least one of them is irrational: There are two tiles rotated against each other by an angle $\theta \notin \pi\mathbb{Q}$. By definition of a substitution tiling, there is a super-tile $\sigma^n(T)$ containing these two tiles, and the claim follows. □

4. The Pythia substitutions

First, consider the following family of substitutions (the Pythagoras substitutions), where the corresponding tilings are *not* pinwheel-like. Let $m \geq 3$ and M be the $(m \times m)$-matrix with entries $M_{i+1,i} = 1$ for $1 \leq i \leq m-1$; $M_{1,m} = 1$; $M_{j+1,m} = 1$ for some $1 \leq j \leq m-1$, and $M_{i,j} = 0$ else. In other words, let M be the companion matrix of the polynomial $p := x^m - x^j - 1$. For simplicity, we require $\gcd(m,j) = 1$. Then, the primitivity of M can be shown easily by the methods in [SE]. (If $\gcd(m, j-1) = q > 1$, the substitution is no longer primitive, but all tilings occurring are already contained in the primitive case.) Let η be the Perron eigenvalue of M. By the remark after Definition 2.2, $\lambda := \sqrt{\eta}$ is the substitution factor of any self-similar substitution with substitution matrix M.

Let $a := \lambda^{-m}$, and T_1 be the orthogonal triangle with vertices $(0,0)$, $(-a,0)$, $(-a,\sqrt{1-a^2})$, so its hypotenuse is of length 1, see Figure 2. This is the first prototile. The other prototiles are $T_{i+1} := \lambda^i T_1$ for $1 \leq i \leq m-1$. The substitution is defined by $\sigma_{m,j}(T_i) = T_{i+1}$ for $1 \leq i \leq m-1$, and $\sigma_{m,j}(T_m) = \{\varphi(T_1), \psi(T_{j+1})\}$. As indicated in Figure 2, the substitution acts on T_m by dissecting λT_m along the altitude on the hypotenuse into tiles of type T_1, T_{j+1}. The edge labels in the figure indicate the edge lengths in terms of powers of λ. E.g., $m-1$ is to be read as λ^{m-1}. Since $\lambda^{2m} = \lambda^{2j} + 1$, it follows $1 = (\lambda^{j-m})^2 + (\lambda^{-m})^2$ and $\lambda^m = \lambda^{j-m+j} + \lambda^{-m}$. The former shows that the triangles are indeed orthogonal triangles, and inspired the name. The latter means that the altitude on the hypotenuse indeed dissects λT_m into T_1 and T_{j+1}. So the Pythagoras substitution $\sigma_{m,j}$ is well-defined.

In order to obtain a pinwheel-like substitution, consider $(\sigma_{m,j})^{2m}(T_1)$ (see Figure 3, left). One diagonal of the grey rectangle, consisting of two tiles T_{j+1}, is the altitude on the hypotenuse of the large triangle. The new substitution $\varrho_{m,j}$ arises by choosing the other diagonal of the rectangle, Figure 3 (centre). This defines the substitution $\varrho_{m,j}(T_1)$ of T_1. The concrete maps used by $\varrho_{m,j}(T_1)$ can be obtained from the figure and the corresponding maps in $\sigma_{m,j}$. The

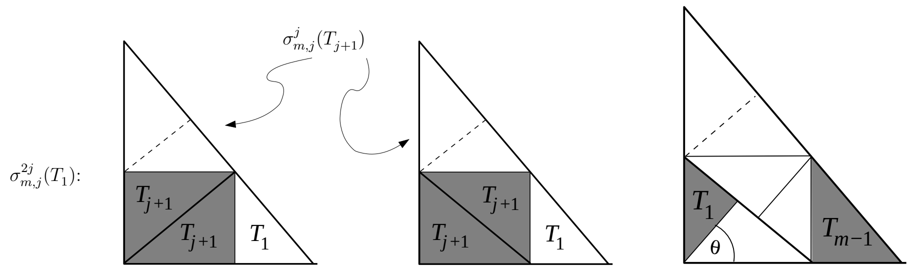

FIGURE 3. The construction of the Pythia substitution $\varrho_{m,j}$ out of the Pythagoras substitution $\sigma_{m,j}$. Left: the $2j$-th iterate of T_1 under $\sigma_{m,j}$. Centre: The same, with two tiles flipped. This patch defines the first iterate of T_1 under $\varrho_{m,j}$. Right: The first iterate of T_{m-j+1}, used in the proof of Theorem 6.2.

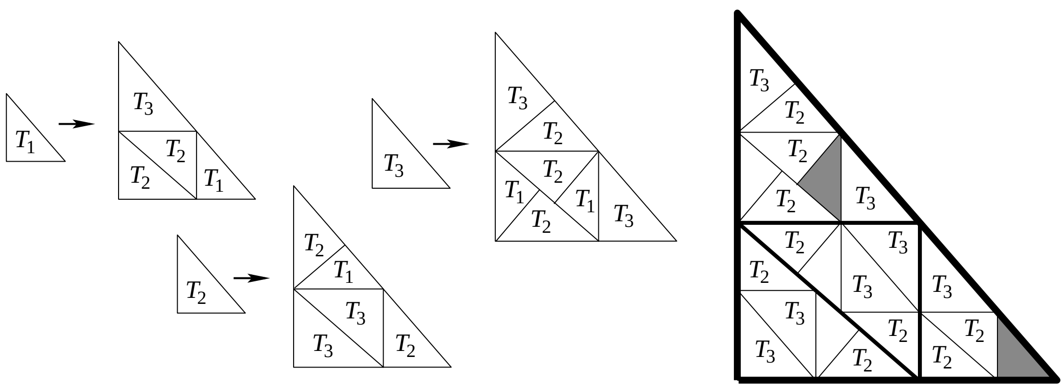

FIGURE 4. The substitution rule for the Pythia substitution $\varrho_{3,1}$ (left), and the second iterate of T_1 (right). The two grey shaded tiles are both of type T_1, rotated against each other by an angle $\theta \notin \pi\mathbb{Q}$.

substitution for the other prototiles is defined by

$$\varrho_{m,j}(T_i) := (\sigma_{m,j})^{i-1}\big(\varrho_{m,j}(T_1)\big), \quad 2 \le i \le m. \tag{2}$$

Since $\sigma_{m,j}$ is well-defined, $\varrho_{m,j}$ is, too. Since the substitution factor of the Pythagoras substitution $\sigma_{m,j}$ is λ, and the Pythia substitution arises from $2m$ iterations of $\sigma_{m,j}$, the substitution factor of the Pythia substitution $\varrho_{m,j}$ is λ^{2m}. The particular Pythia substitution $\varrho_{3,1}$ is shown in Figure 4. The proof that all Pythia substitutions $\varrho_{m,j}$ generate pinwheel-like tilings is given in Theorem 6.2.

5. **The tipi substitutions**

Let $\eta_{m,j}$ be the root of $x^m - x^{2j} - 2x^j - 1$, $3 \le m$, $1 \le j < m/2$, $a = a_{m,j} = (\eta_{m,j})^j$, $\theta := \arccos(1/2a)$. As above, m is always the number of prototiles of the tiling under consideration. The first prototile, T_0, is an isosceles triangle with edges of length $1, a, a$. To be explicit, let T_0 be the triangle with vertices 0, 1, $\frac{1}{2} + i\sqrt{a^2 - \frac{1}{4}}$. The other prototiles are defined recursively by $T_i := \eta_{m,j} T_{i-1}$ $(1 \le i \le m-1)$. The substitution rule is $\sigma(T_i) := T_{i+1}$ for $0 \le i \le m-2$,

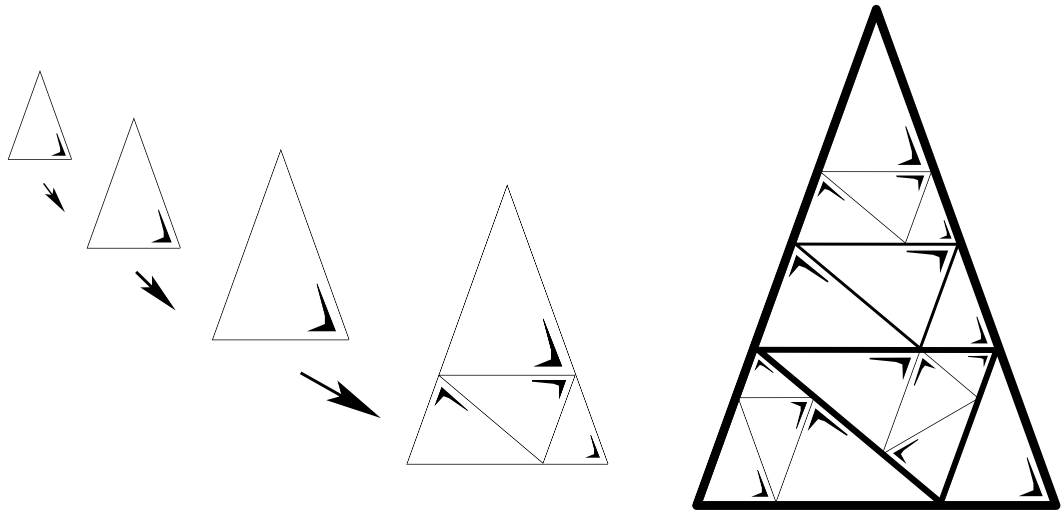

FIGURE 5. The tipi substitution for the case $m = 3$, $j = 1$. The shape of the tiles resembles a tipi, a conical tent used by Sioux and other tribes in the Great Plains, thus the name.

and $\sigma(T_m) := \{\varphi_0(T_0), \varphi_1(T_j), \varphi_2(T_j), \varphi_3(T_{2j})\}$; see Figure 4 for the case $j = 1$, $m = 3$. The Euclidean motions φ_i can be read off from the figure. Explicitly, they read

$$\varphi_0(z) = z + a^2,\ \varphi_1(z) = e^{2\pi i\theta} z + a^2,\ \varphi_2(z) = e^{2\pi i\theta}\overline{z}, \varphi_3(z) = z + \frac{1}{2} + i\sqrt{a^2 - \frac{1}{4}}.$$

Note that one reflection is involved, expressed by the complex conjugation in φ_2.

6. **Properties of the Tilings**

The uniform distribution of orientations was shown for the particular case of the pinwheel tiling in [RA2], see also [MPS]. The present proof is just a generalisation to the arbitrary case. It uses Perron's theorem [PER], Weyl's criterion and Proposition 3.4.

Theorem 6.1. *Let $\mathcal{T}_\sigma$ be a pinwheel-like substitution tiling, where σ is a primitive substitution in $\mathbb{R}^2$ with prototiles $T_1, \ldots, T_m$. Then $\mathcal{T}_\sigma$ is of statistical circular symmetry. Consequently, $\mathbb{X}_\sigma$ is of statistical circular symmetry, too.*

Proof. Weyl's criterion states that $\{\alpha_j\}_{j\in\mathbb{Z}} = \{e^{i\varphi_j}\}_{j\in\mathbb{Z}}$ is uniformly distributed in $[0, 2\pi[$, iff

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} (\alpha_j)^t = 0$$

for all $0 \neq t \in \mathbb{Z}$. So, for $t \in \mathbb{Z}$, consider the matrix

$$M(t) = \left(\sum_{j=1}^{S_{k\ell}} e^{i\varphi_j t} \right)_{k\ell} \quad (1 \leq k, \ell \leq m),$$

where S denotes the substitution matrix and, for k, ℓ given, φ_j denotes the orientation of the j-th tile of type k in $\sigma(T_\ell)$. Thus, $\big(M(1)\big)_{k\ell}$ contains the sum of the orientations of the tiles

of type k in $\sigma(T_\ell)$. Inductively, it follows that $\big(M(1)^r\big)_{k\ell}$ contains the sum of the orientations of the tiles of type k in $\sigma^r(T_\ell)$. We proceed to prove

$$\lim_{r\to\infty} \frac{(M(t)^r)_{k\ell}}{(S^r)_{k\ell}} = 0,$$

from which uniform distribution follows. Without loss of generality, let one occurring orientation be 0. (The substitution rule can always be modified in order to achieve that, by adding an appropriate general rotation.) Let $r \geq 1$. Assume

$$|\big(M(t)^r\big)_{k\ell}| = |\sum_{j=1}^{(S^r)_{k\ell}} e^{i\varphi_j t}| = S_{k\ell},$$

for some $t > 0$. Then, $\varphi_j t \in \pi\mathbb{Z}$ for all φ_j considered (that is, for all φ_j associated with some T_k in $\sigma^r(T_\ell)$). But, by Proposition 3.4, not all occurring angles φ_j are elements of $\pi\mathbb{Q}$. Thus, for all $r, t \geq 1$, there is (k,ℓ) such that

$$|\sum_{j=1}^{(S^r)_{k\ell}} e^{i\varphi_j t}| < (S^r)_{k\ell}.$$

Let λ be the substitution factor of the substitution (so, λ^2 is the Perron-Frobenius eigenvalue of S), and let η be the Perron-Frobenius eigenvector of $A := \big(|(M(t)^r)_{k\ell}|\big)_{k\ell}$. By Perron's theorem, $0 < \lim_{r\to\infty} \frac{(S^r)_{k\ell}}{(\lambda^2)^r} = c_{k\ell}$, with a constant $0 < c_{k\ell} < \infty$. (More precisely, each vector $(c_{k1}, \ldots, c_{km})^T$ is a right eigenvector of S for the Perron-Frobenius eigenvalue λ^2 of S, and each vector $(c_{1\ell}, \ldots, c_{m\ell})$ is a left eigenvector for λ^2, and all vectors are positive.) Analogously, $0 < \lim_{r\to\infty} \frac{(A^r)_{k\ell}}{\eta^r} = a_{k\ell} < \infty$. Since $A \leq S^r$ and $A \neq S^r$, it follows $\eta < \lambda^2$, again by Perron's theorem. Thus,

$$\left|\frac{(M(t)^r)_{k\ell}}{(S^r)_{k\ell}}\right| \leq \frac{(A^r)_{k\ell}}{(S^r)_{k\ell}} = \frac{(A^r)_{k\ell}}{\eta^r}\frac{\eta^r}{(S^r)_{k\ell}} \leq c\left(\frac{\eta}{\lambda^2}\right)^r \to 0 \quad (r\to\infty).$$

The claim for the hull $\mathbb{X}_\sigma$ follows immediately, since each element of $\mathbb{X}_\sigma$ shows already statistical circular symmetry. □

Theorem 6.2. *Let $m \geq 3$, and $1 < j < m$. All tilings defined by the Pythia substitution $\varrho_{m,j}$ are of statistical circular symmetry.*

Proof. Utilising Proposition 3.4, we need to show that there are two tiles of the same type T_i in some $(\varrho_{m,j})^k(T_j)$ which are rotated against each other by an angle $\theta \notin \pi\mathbb{Q}$.

Consider the Pythia substitution with respect to chirality of the tiles. Let us call the tiles in Figure 2 (without the two rightmost ones) right-handed, denoted by T_i as usual; and let us call their mirror images — obtained by reflection in the horizontal axis — left-handed, denoted $\overline{T}_i$. The following properties follow from the construction of the Pythia substitution $\varrho_{m,j}$.

1. For each T_i, $\varrho_{m,j}(T_i)$ contains a translate of T_i, rotated by $0 \mod \frac{\pi}{2}$.
2. $\varrho_{m,j}(T_1)$ contains a translate of $\overline{T}_{j+1}$, rotated by $0 \mod \frac{\pi}{2}$.
3. $\varrho_{m,j}(T_{m-j+1})$ contains a translate of T_1, rotated by $\theta \mod \frac{\pi}{2}$.
4. $\varrho_{m,j}(T_{m-j+1})$ contains a translate of $\overline{T}_{j+1}$, rotated by $-\theta \mod \frac{\pi}{2}$.

The first three statements are immediate consequences of the construction, see Figure 3. For the fourth one, recall that the m-th iterate of T_1 under the Pythagoras substitution $(\sigma_{m,j})^m$ contains a translate of $\overline{T}_1$, rotated by $-\theta \mod \frac{\pi}{2}$, see Figure 2. Consequently, $(\sigma_{m,j})^m(T_{j+1})$ contains a translate of $\overline{T}_{j+1}$, rotated by $-\theta \mod \frac{\pi}{2}$. By definition, $\varrho_{m,j}(T_{m-j+1})$ contains a translate of $(\sigma_{m,j})^{j+m-j}(T_{j+1}) = (\sigma_{m,j})^m(T_{j+1})$ (compare Figure 3 and (2)), thus it contains a translate of $\overline{T}_{j+1}$, rotated by $-\theta \mod \frac{\pi}{2}$.

Altogether, by 3. and 2., $(\varrho_{m,j})^2(T_{m-j+1})$ contains a translate of $\overline{T}_{j+1}$, rotated by $\theta \mod \frac{\pi}{2}$. By 4. and 1., $(\varrho_{m,j})^2(T_{m-j+1})$ contains a translate of $\overline{T}_{j+1}$, rotated by $-\theta \mod \frac{\pi}{2}$. By Proposition 3.4, it remains to show that $\theta - (-\theta) = 2\theta \notin \pi\mathbb{Q}$.

We proceed by showing that $\theta \notin \pi\mathbb{Q}$. Consider the tile T_1 embedded in $\mathbb{C}$, with vertices $0,\ -\lambda^{-m},\ -\lambda^{-m} + i\lambda^{j-m}$, see Figure 2. Then $\theta \notin \pi\mathbb{Q}$ iff $-\lambda^{-m} + i\lambda^{j-m}$ is not a complex root of unity, or equivalently, $\lambda^{-m} \neq \cos(\frac{k\pi}{n})$ for all $k, n \in \mathbb{Z}$. Recall that λ^2 is a root of $x^m - x^j - 1$, so λ is a root of $p := x^{2m} - x^{2j} - 1$. It is well known that a polynomial in $\mathbb{Z}[x]$ which is reducible in $\mathbb{Q}(x)$ is already reducible in $\mathbb{Z}[x]$. Thus, no non-integer coefficients can occur in the factorisation of p, wherefore the prime polynomial of λ is of the form $x^\ell \pm \cdots \pm 1$. Consequently, λ is an algebraic integer, as well as a unit. Consequently, λ^{-1} is an algebraic integer as well, as is λ^{-m}.

Assume that $\lambda^{-m} = \cos(\frac{k\pi}{n})$ for some $k, n \in \mathbb{Z}$, where $\gcd(k, n) = 1$. Since $\lambda^{-m} \notin \{-1, 0, 1\}$, we can exclude $k = 0,\ n = 1$ and $n = 2$. Let $\xi_n = e^{2\pi i \frac{k}{n}}$ be a primitive n-th root of unity. Then, $\lambda^{-m} = \frac{1}{2}(\xi_n + \overline{\xi_n})$.

On the other hand, it is known that the ring of integers in $\mathbb{Q}(\xi_n + \overline{\xi_n})$ equals $\mathbb{Z}[\xi_n + \overline{\xi_n}]$ [W, Prop. 2.16], and all integers in $\mathbb{Q}(\xi_n + \overline{\xi_n})$ are of the form $\sum_{k=0}^{n-1} \beta_k(\xi_n + \overline{\xi_n})^k$, where $\beta_k \in \mathbb{Z}$. But the unique representation of λ^{-m} is $\frac{1}{2}(\xi_n + \overline{\xi_n})$, so λ^{-m} is not an algebraic integer, which is a contradiction. Therefore, $\lambda^{-m} \neq \cos(\frac{k\pi}{n})$ for all $k, n \in \mathbb{Z}$, which proves the claim. $\square$

Note that the relevant angle for the tipi substitution is φ with $\cos(\varphi) = \frac{1}{2}(\eta_{m,k})^{-j}$, and $(\eta_{m,k})^m = (\eta_{m,k})^j + 1$. So the above argument fails in this case: it may be that $\eta_{m,k} = 2\cos(\frac{k\pi}{n})$. We don't see a similarly simple argument for all tipi tilings being of statistical circular symmetry. Nevertheless, each particular case can be easily checked whether it is. All cases checked so far are of statistical circular symmetry.

Theorem 6.3. *Every primitive substitution tiling $\mathcal{T}_\sigma$ of statistical circular symmetry shows uniform patch frequency, in the following sense: For every $\varepsilon > 0$ and each finite patch $P \subset \mathcal{T}$ there is $r > 0$ such that every ball of radius r contains a translate of P, up to a small rotation R_α, where $|\alpha| < \varepsilon$.*

Proof. Let $\varepsilon > 0$, and let P be some patch in $\mathcal{T}$. By Definition 2.2 there is $k \geq 1$ such that a copy of P is contained in some supertile $\sigma^k(T_i)$. By primitivity, there is $M > k$ such that a copy of P is contained in all supertiles $\sigma^m(T)$, $m \geq M$.

Let T_j be some prototile. By the proof of Proposition 3.4, there is $\theta \notin \pi\mathbb{Q}$ such that holds: For all $N \geq 0$ there is $\ell \geq 0$ such that each supertile $\sigma^\ell(T)$ contains a translate of $R_{n\theta}T_j$ for all $0 \leq n \leq N$.

Every angle $\beta \in [0, 2\pi[$ can be approximated by $n\theta \mod \pi$ (with suitable $n \geq 0$) arbitrarily close.

Let us combine these observations. There is a particular supertile $\sigma^\ell(T_i)$ containing translates of $R_{n\theta}T_j$ for all $0 \leq n \leq N$. Thus, there is $s > 0$ such that all supertiles $\sigma^{\ell+s}(T)$ contain translates of $R_{n\theta}T_j$ for all $0 \leq n \leq N$. Then, all supertiles $\sigma^{\ell+s+m}(T)$ contain copies of P, in angles $0, \theta, \dots, N\theta$. The orientation of P can be approximated arbitrarily close by increasing m. Thus, there are ℓ, s, m such that each supertile $\sigma^{\ell+s+m}(T)$ contains a copy P' of P, with $P' = R_\alpha P - t$, $t \in \mathbb{R}^2$, $|\alpha| < \varepsilon$. For this particular choice ℓ, s, m, there is $r' > 0$ such that each supertile $\sigma^{\ell+s+m}(T)$ fits into a ball with radius r'. Thus each ball of radius $r = 3r'$ contains an entire supertile, which proves the claim. $\square$

7. Remarks

The circular symmetry of $\mathbb{X}_\sigma$ implies the circular symmetry of the autocorrelation (compare for instance [MPS], [BFG]) of its elements, and therefore the circular symmetry of its diffraction spectrum. This has been known for the pinwheel tilings. The results in this article imply the circular symmetry of the diffraction of all primitive substitution tilings with tiles in infinitely many orientations.

The relevance of Theorem 6.3 relies on the connection to dynamical systems of a tiling space. In the case of finitely many orientations (that is, the number of $\alpha(T)$ as in (1) is finite), the dynamical system $(\mathbb{X}_\sigma, \mathbb{R}^2)$ is uniquely ergodic, if and only if the tilings in $\mathbb{X}_\sigma$ have uniform patch frequency [LMS]. This result plays a central role for further investigations of such systems. The generalization of this result to tilings with statistical circular symmetry will be worth investigating in the future.

A tiling is of *finite local complexity* (FLC), if for each $r > 0$, the number of congruence classes of $\mathcal{T} \cap B_r(x)$ $(x \in \mathbb{R}^d)$ is finite. The Pythia and the tipi substitution tilings introduced here show both cases: FLC and non-FLC. This can be shown by the methods in [Dan2], [FrR]. Note that, in the case of finitely many orientations, FLC is frequently defined by 'for each $r > 0$, the number of *translation* classes of $\mathcal{T} \cap B_r(x)$ $(x \in \mathbb{R}^d)$ is finite'. In the present context we deviate from this convention for obvious reasons.

The substitution factor of a self-similar substitution determines already many properties of the corresponding tilings. For instance, if the substitution factor is a PV number, then the tiling is FLC under a certain (mild) condition [FrR]. PV number stands for Pisot-Vijayaraghavan number, an algebraic integer which algebraic conjugates are all smaller than one in modulus. The substitution factors of the tilings in this article cover various cases. In particular, they are of arbitrary algebraic degree $m \geq 2$. For $m \geq 3$, this can be seen by the irreducibility of $x^m - x - 1$. For $m = 2$, one needs to alter the setting slightly: The case $m = 4, j = 2$ does not obey the requirement $\gcd(m, j) = 1$. Therefore, the Pythagoras substitution for these values is not longer primitive: there are four prototiles T_1, T_2, T_3, T_4, but in each Pythagoras tiling to $\sigma_{4,2}$, only two of them occur, either T_1, T_3, or T_2, T_4. The substitution $(\sigma_{4,2})^2$ does the job: It uses only two prototiles T_1, T_3. The substitution factor is the golden mean $\tau = \frac{\sqrt{5}+1}{2}$. The corresponding Pythia tiling (see 'golden pinwheel tilings' in [FH]) has substitution factor $\tau + 1$, which is a quadratic PV number. Other PV numbers occurring as substitution factors for Pythagoras tilings are the dominant roots of $x^3 - x - 1$ (which is the smallest PV number among all algebraic integers [Sie]), of $x^3 - x^2 - 1$ and of $x^4 - x^3 - 1$. Thus, PV numbers which

are substitution factors of the Pythia tilings include powers of those. PV numbers occurring as substitution factors for tipi tilings include the ones mentioned for the Pythagoras tilings, as well as the dominant root of $x^3 - 2x^2 + x - 1$ (in the case $m = 5, j = 2$).

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